JGP - Vol. 3, n. 3, 1986

The Maslov class of some Legendre submanifolds

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Abstract. *In the paper, by using a differential-geometric machinery, one computes the Maslov class for: a) Legendre curves on S^3, with respect to any one of the three classical contact forms of S^3; b) Legendre submanifolds for the classical contact structure of the cotangent unit spheres bundles of a Riemannian manifold N. In case b), and if N is flat, the Maslov class is determined by the mean curvature vector, and it vanishes if the Legendre submanifold is minimal.*

1. INTRODUCTION

The *Maslov class* in an important invariant in symplectic-Lagrangian geometry, and its applications to mathematical physics. Generally, this is a 1-dimensional cohomology class $m \in H^1(M, \mathbb{R})$, associated to a symplectic vector bundle $p : E \to M$, and two Lagrangian subbundles L_1, L_2 of the former, and it is an obstruction to the transversality of L_1 and L_2. If γ is a closed curve in M, then $\int_\gamma m$ is the *Maslov index* of γ. In the particular case of a cotangent bundle T^*N, the Maslov class and index appear for Lagrangian submanifolds $M \subset T^*N$, by taking $E = T(T^*N)/M$, $L_1 = V/M$, $L_2 = TM$, where V is tangent to the fibers of T^*N, and they play a fundamental role in constructing asymptotic solutions of differential operators on N by the so-called method of the canonical operator. We refer the reader to [GS] for a general discussion of this subject, and for further references to the relevant literature.

Key-Words: *Maslov class, Legendre submanifolds, Contact geometry.*
1980 Mathematics Subject Classification: 58 F 05.

The original definition of the Maslov class (index) ammounts to the consideration of the degree of a certain mapping of the circle S^1, but among the methods developed for its computation some are based on differential-geometric machineries [D], [M]. Particularly, it has been remarked by Kamber and Tondeur [KT] that the Maslov class can be interpreted as an exotic characteristic class, and in our paper [V_2], we have developed this viewpoint by discussing also higher dimensional Maslov classes and by giving computational formulas by means of connections.

While the definition of the Maslov classes is more general, computations of these classes were done mainly for Lagrangian submanifolds of cotangent bundles (mostly of $T^*\mathbb{R}^n = \mathbb{R}^{2n}$). It is the aim of the present note to give some other geometrically interesting examples of a computation of a Maslov class, and we describe them shortly here.

The examples are taken from contact geometry. If V^{2n+1} is a contact manifold with the contact 1-form η [Bl], then, the distribution $\eta = 0$ is a symplectic vector bundle E of rank $2n$ over M, with the symplectic form $d\eta$. The maximal dimension of integral submanifolds of E is n, and an n-dimensional integral submanifold of E is called a *Legendre submanifold* of V.

Let us assume that we have: a) a Legendre submanifold M of V, and b) a Lagrangian subbundle C of E. Then $(E/M, C/M, TM)$ is a configuration for which Maslov classes may be defined. A first example is provided by the 3-dimensional unit sphere. S^3 has three contact structures, whose associated canonical vector fields ξ_a ($a = 1, 2, 3$) define the well known parallelization of S^3, and any curve γ tangent to the plane (ξ_2, ξ_3) is a *Legendre curve* with respect to the contact structure of ξ_1 (and so on) [Bl]. We shall see that it is easy to compute the Maslov class (index) of such Legendre curves γ with respect to the foliation $\mathcal{C}$ defined by the orbits of (say) ξ_2.

The main example to be considered is the general situation obtained by looking at the cotangent unit spheres bundle S^*N over a Riemannian manifold N. S^*N has a natural contact structure induced by the Liouville 1-form of the cotangent bundle T^*N [Bl]. The fibers of S^*N define a Legendre foliation $\mathcal{C}$, and it is natural to look for the Maslov class of an arbitrary Legendre submanifold M of S^*N with respect to $\mathcal{C}$. In analogy with a result of J.M. Morvan [M], we shall express this class by means of the mean curvature vector of M.

Needless to say, in this paper we are working in the C^∞ category. Another convention which we constantly use is Einstein's summation convention.

2. GENERAL FORMULAS

Let us start by formulating one of the possible definitions of the Maslov class of a triple (E, L_1, L_2) where E is a symplectic vector bundle with the symplectic vector bundle with the symplectic form Ω over a manifold M, and L_1, L_2 are Lagrangian subbundles of the former. First, we have to choose on E a complex structure J compatible with the symplectic structure (i.e., $\Omega(Ja, Jb) = \Omega(a, b)$). Then, if $U \subset M$ is a trivializing neighbourhood for our bundles, we may look at fields of frames (e_α^U), (f_α^U) $(\alpha = 1, \ldots, 2n)$ of the fibers of E/U which are unitary for the Hermitian metric determined by $g(a, b) = \Omega(a, Jb)$, and are such that $e_i^U \in L_1$, $f_i^U \in L_2$ $(i = 1, \ldots, n)$. Accordingly, we have complex local bases for the complex n-dimensional vector bundle (E, J) given by

$$\begin{aligned} \epsilon_i^U &= \frac{1}{\sqrt{2}} (e_i^U - \sqrt{-1}\, J\, e_i^U) \\ \varphi_i^U &= \frac{1}{\sqrt{2}} (f_i^U - \sqrt{-1}\, J\, f_i^U), \end{aligned} \tag{2.1}$$

and the transition between these bases is of the form

$$\varphi^U = A_U \epsilon^U \tag{2.2}$$

where $A_U : U \to U(n) =$ the unitary n-dimensional group, and a change in the choice of the fields of frames multiplies A_U at the left and at the right by matrices belonging to the orthogonal group $0(n)$. Now, we see that for a covering of M with such neighbourhoods U, the local mappings A_U glue up to a global well defined set mapping

$$A(L_1, L_2) : M \to U(n) /\!\!/ 0(n), \tag{2.3}$$

where $/\!\!/$ denotes equivalence of unitary matrices by both left and right multiplication by orthogonal matrices.

Furthermore, since the determinant of an orthogonal matrix is ± 1, there is a well defined mapping

$$\det{}^2 : U(n) /\!\!/ 0(n) \to S^1 = \{z \in \mathbb{C} / |z| = 1\} \tag{2.4}$$

defined by the square of the determinant of a matrix.

Hence, we obtain a mapping

$$\varphi_{L_1 L_2} = \det{}^2 \circ A(L_1, L_2) : M \to S^1, \tag{2.5}$$

which is clearly differentiable, and the *Maslov class* is defined as

$$m(L_1, L_2) = \varphi_{L_1 L_2}^* \left[\frac{\mathrm{d}z}{2\pi \sqrt{-1}\, z} \right] \in H^1(M, \mathbb{R}), \tag{2.6}$$

where $\mathrm{d}z/2\pi\sqrt{-1}\,z$ is the «volume form» of S^1 of (2.4), and brackets denote cohomology classes.

The Maslov class does not depend on the choice of J since any two such operators J are homotopically related, and it vanishes if L_1, L_2 are transversal subbundles of E since then J may be chosen such that $L_2 = JL_1$. If $\gamma : S^1 \to M$ is a closed curve in M, then the *Maslov index* of γ is defined by

$$(2.7)\qquad m^{\gamma}_{L_1L_2} = \int_{\gamma} m(L_1, L_2) = \deg(\varphi_{L_1L_2} \circ \gamma),$$

where in the right-hand side we have the degree of the corresponding mapping, and we see that $m^{\gamma}_{L_1L_2}$ is an integer.

Let us also remember the general definition of Maslov classes as exotic characteristic classes [V$_2$]. Let us look again at the bundles (E, L_1, L_2) and at the compatible complex structure J considered above. Let us also look at a covering of M by trivialization neighbourhoods U, and at local frame fields ϵ_U, φ_U as in formula (2.1). Using such frames, we may define two kinds of linear connections on E namely, L_1*-orthogonal connections*, and L_2*-orthogonal connections*. In a precise manner, (e^i_U) $(i = 1, \dots, n)$ is a g-orthonormal basis of L_1, and there are corresponding orthogonal connections in L_1 given locally by

$$(2.8)\qquad De_i = \omega^j_i\, e_j \qquad \omega^j_i + \omega^i_j = 0.$$

Then, formula (2.1) yields complex bases of (E, J), and

$$(2.9)\qquad D\epsilon_i = \omega^j_i\, \epsilon_j$$

is a connection in (E, J) which we call L_1-orthogonal. The L_2-orthogonal connections will be defined similarly be means of formulas

$$(2.10)\qquad D\varphi_i = \tilde{\omega}^j_i \varphi_j \qquad (\tilde{\omega}^j_i + \tilde{\omega}^i_j = 0),$$

where φ_i are defined in (2.1). (It is easy to understand the meaning of these types of connections in terms of principal bundles of frames and, hence, the existence of such connections).

Now, the existence of these two kinds of connections leads to exotic charactesitic classes by means of the known comparison procedure due to Bott [Bo]. Namely, let us consider the *Chern polynomials*

$$(2.11)\qquad c_k(A) = \left(\frac{\sqrt{-1}}{2\pi}\right)^k \operatorname{tr} \Lambda^k A,$$

where $A \in u(n)$ = the unitary Lie algebra, and $\Lambda^k A$ denotes the k^{th}-compound of A. Let $\overset{1}{\nabla}$ be a L_1-orthogonal connection given by (2.9), and $\overset{2}{\nabla}$ be a L_2-orthogonal connection given by (2.10), and denote by $(\overset{1}{\pi}{}^j_i)$, $(\overset{2}{\pi}{}^j_i)$ the matrices of these connections in a common complex unitary basis (b_i) of (E, J) (notice that in (2.9), (2.10), we had different bases for the two connections), and by $(\overset{1}{\Pi}{}^j_i)$, $(\overset{2}{\Pi}{}^j_i)$ the corresponding curvature matrices. Then, the c_k*-difference form* is defined by

$$
\begin{aligned}
\Delta_{12}c_k &= k\int_0^1 c_k(\overset{2}{\pi}-\overset{1}{\pi}, \underbrace{\overset{t}{\Pi}, \ldots, \overset{t}{\Pi}}_{(k-1)\,\text{times}})\,\mathrm{d}t = \\
&= \frac{(\sqrt{-1})^k}{(2\pi)^k(k-1)!}\int_0^1 \left\{\delta^{j_1\ldots j_k}_{i_1\ldots i_k}(\overset{2}{\pi}{}^{i_1}_{j_1}-\overset{1}{\pi}{}^{i_1}_{j_1})\wedge\overset{t}{\Pi}{}^{i_2}_{j_2}\wedge\ldots\wedge\overset{t}{\Pi}{}^{i_k}_{j_k}\right\}\mathrm{d}t,
\end{aligned}
\tag{2.12}
$$

where $\overset{t}{\Pi} = (\overset{t}{\Pi}{}^j_i)$ is the curvature matrix of the connection $(1-t)\overset{1}{\nabla} + t\overset{2}{\nabla}$ $(0 \leqslant t \leqslant 1)$. The basic property of the form (2.12) is

$$
\mathrm{d}\Delta_{12}c_k = c_k(\overset{2}{\Pi}) - c_k(\overset{1}{\Pi}), \tag{2.13}
$$

and it is known that for odd $k = 2h-1$ the right-hand side of (2.13) vanishes. (See, for instance, [Bo]).

Accordingly, we obtain the cohomology classes

$$
\mu^h_{L_1L_2} = [\Delta_{12}c_{2h-1}] \in H^{4h-3}(M, \mathbb{R}), \tag{2.14}
$$

and we call (2.14) the h^{th}*-Maslov class of* (L_1, L_2) $(h = 1, 2, \ldots)$. These classes depend neither on the choice of J nor on that of the orthogonal connections, and they vanish if L_1 and L_2 are transversal subbundles, i.e., all these classes are transversality obstructions [V$_2$].

Particularly, the first class $\mu^1_{L_1L_2}$ is represented by

$$
\Delta_{12}c_1 = \frac{\sqrt{-1}}{2\pi}(\overset{2}{\pi}{}^i_i - \overset{1}{\pi}{}^i_i), \tag{2.15}
$$

and the important fact for the present note is [KT], [D], [V$_2$]

$$
\mu^1_{L_1L_2} = \frac{1}{2}m(L_1, L_2). \tag{2.16}
$$

It is on these formulas that we base the computation of the Maslov class in the next section.

3. COMPUTATIONS OF MASLOV CLASSES

In this section we develop the examples described in Introduction.

We shall start with the simple case of curves in S^3. Let us identify $\mathbb{R}^4$ with the field Q of the quaternions by means of the mapping

$$(x^1, x^2, x^3, x^4) \mapsto q = x^1 + x^2 i + x^3 j + x^4 k, \tag{3.1}$$

where $(1, i, j, k)$ is the usual basis of Q. Then

$$S^3 = \{q \in Q / q\bar{q} = 1\}, \tag{3.2}$$

and it follows that the vector fields

$$\begin{aligned} \xi_1 &= iq = (-x^2, x^1, -x^4, x^3) \\ \xi_2 &= jq = (-x^3, x^4, x^1, -x^2) \\ \xi_3 &= kq = (-x^4, -x^3, x^2, x^1) \end{aligned} \tag{3.3}$$

are in $T_q S^3$, and define a field of orthonormal tangent frames on S^3 with the metric induced by $\mathbb{R}^4$.

Let us denote by (η_1, η_2, η_3) the dual cobasis of (3.3). Then, for instance

$$\eta_1 = -x^2 \mathrm{d}x^1 + x^1 \mathrm{d}x^2 - x^4 \mathrm{d}x^3 + x^3 \mathrm{d}x^4, \tag{3.4}$$

and it is easy to see that this is a contact form on S^3 [Bl]. One can write down similarly the contact forms η_2, η_3. Clearly, the contact distribution E_1 of (3.4) is spanned by $\{\xi_2, \xi_3\}$, and $\mathrm{d}\eta_1$ defines on E_1 the structure of a symplectic vector bundle of rank 2. Any curve γ of S^3 which is tangent to E_1 will be called a ξ_1--Legendre curve. (Of course, we have similarly ξ_2-Legendre curves and ξ_3-Legendre curves).

Now, following the explanation given in Introduction, let us also look at the foliation C_2 of S^3 by orbits of ξ_2, which is obviously a ξ_1-Legendre distribution. Then, if γ is a ξ_1-Legendre curve on S^3 we have along γ the symplectic vector bundle E_1/γ, and the Lagrangian subbundles $L_1 = \operatorname{span} \xi_2/\gamma$, $L_2 = \operatorname{span} \dot{\gamma}$ (where the dot denotes derivative with respect to arc length), and the class $m(\gamma) = m(L_1, L_2)$ will be called the *Maslov class* of γ. If γ is closed $\int_\gamma m(\gamma)$ is the *Maslov index* of γ.

We can compute $m(\gamma)$ straightforwardly from the first definition given in Section 2, without using connections. First, we shall notice that

$$J\xi_2 = \xi_3, \qquad J\xi_3 = -\xi_2 \tag{3.5}$$

defines on E a complex structure compatible with the symplectic form $\Omega = \mathrm{d}\eta_1/E$, and that the associated Hermitian metric is precisely the one induced by the Euclidean metric of $\mathbb{R}^4$.

Now, the index i of (2.1) takes only one value, and we may use the bases (2.1) with

$$e_1 = \xi_2, \qquad \epsilon_1 = \frac{1}{\sqrt{2}}(\xi_2 - \sqrt{-1}\,\xi_3),$$
(3.6)
$$f_1 = \dot{\gamma}, \qquad \varphi_1 = \frac{1}{\sqrt{2}}(\dot{\gamma} - \sqrt{-1}\,J\dot{\gamma}).$$

If θ denotes the angle of γ with ξ_2 in E, we have

(3.7) $$\dot{\gamma} = \xi_2 \cos\theta + \xi_3 \sin\theta, \qquad J\dot{\gamma} = -\xi_2 \sin\theta + \xi_3 \cos\theta,$$

and accordingly

(3.8) $$\varphi_1 = \epsilon_1(\cos\theta + \sqrt{-1}\sin\theta) = e^{\sqrt{-1}\,\theta}\epsilon_1.$$

Hence, the matrix A_U of (2.2) is just $e^{\sqrt{-1}\,\theta}$, and it follows from (2.6) that $m(\gamma)$ is the cohomology class defined by $\frac{1}{\pi}\,\mathrm{d}\theta$. If γ is closed, and we see it as an immersion $\gamma : S^1 \to S^3$, we may formulate this result as

PROPOSITION. *The Maslov index of a ξ_1-Legendre curve of S^3 with respect to ξ_2 is the degree of the mapping $(2\theta)\circ\gamma : S^1 \to S^1$, where θ is seen as an angle mapping.* ■

Now, we go over to the discussion of our main example.

Let N^{n+1} be a Riemannian manifold with the metric u given by $\mathrm{d}s^2 = u_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$; $a, b, \ldots, = 1, \ldots, n+1$; x^a = local coordinates on N. Let ξ_a denote the associated natural covector coordinates. Then the cotangent bundle T^*N has the canonical local coordinates (x^b, ξ_a), and the cotangent unit spheres bundle of N is defined as the submanifold of T^*N given by

(3.9) $$i : S^*N = \{(x, \xi) \in T^*N / u^{ab}(x)\,\xi_a \xi_b = 1\} \subset T^*N.$$

It is classical that T^*N possesses the *Liouville 1-form* $\lambda = \xi_a\,\mathrm{d}x^a$, and this induces a contact 1-form $i^*\lambda = \eta$ on S^*N [B1]. The fibers of S^*N, which are defined by x^a = const., obviously satisfy the conditions $\eta = 0$, $\mathrm{d}\eta = 0$, which, for every submanifold tangent to $\eta = 0$, are equivalent with the integrability of the submanifold [B1]. Hence these fibers define a Legendre foliation which we shall denote by C^* on S^*N. Accordingly, if $j : M \to S^*N$ is an arbitrary (immersed) Legendre submanifold of S^*N, we have as in Introduction E/M (given by $\eta = 0$ with the symplectic form $\mathrm{d}\eta$), $L_1 = C^*/M$, $L_2 = TM$, which give rise to

Maslov classes $m(L_1, L_2) \overset{\text{def}}{=} m^*(M)$, $\mu^h_{L_1L_2} \overset{\text{def}}{=} \mu^*_h(M)$. It is our aim now to compute $m^*(M)$, and get thereby a theorem analogous to the one given by Morvan for Lagrangian submanifolds of T^*M [M], [V_2].

Let us recall the existence of the following structures of T^*N [YI]. The Riemannian connection of N has an associated *horizontal distribution* $\mathcal{H}$ on T^*N, which is defined locally by

$$(3.10) \qquad \theta_a = \mathrm{d}\xi_a - \Gamma^c_{ab}\,\xi_c \mathrm{d}x^b = 0,$$

where $\Gamma^{\cdot}_{\cdot\cdot}$ are Christoffel symbols of u. Thereby, T^*N gets an almost product structure ($\mathcal{H}$, V = the tangent distribution of the fibers of T^*N), $(\mathrm{d}x^a, \theta_a)$ is an adapted cobasis of this structure, and

$$(3.11) \qquad X_a = \frac{\partial}{\partial x^a} + \Gamma^c_{ab}\,\xi_c \frac{\partial}{\partial \xi_b}\,, \ \frac{\partial}{\partial \xi_a}$$

is the corresponding dual basis, where $\mathcal{H} = \text{span}\ X_a$, $V = \text{span}\ \partial/\partial\xi_a$. Now, duality with respect to u yields another basis of $\mathcal{H}$

$$(3.12) \qquad Y^a = u^{ab} X_b,$$

and if we look at the transition relations of these local bases we see that the formulas

$$(3.13) \qquad JY^a = \frac{\partial}{\partial \xi_a}\,, J\frac{\partial}{\partial \xi_a} = -Y^a$$

provide us with a well defined almost complex structure on T^*N. Moreover, the relations

$$(3.14) \qquad g(Y^a, Y^b) = g\left(\frac{\partial}{\partial \xi_a}\,, \frac{\partial}{\partial \xi_b}\right) = u^{ab}, g\left(Y^a, \frac{\partial}{\partial \xi_b}\right) = 0$$

yield a J-Hermitian metric with $V \perp \mathcal{H}$, and whose Kähler form $\Omega(X, Y) = g(JX, Y)$ is precisely the canonical symplectic form of T^*M

$$(3.15) \qquad \Omega = -\mathrm{d}\lambda = \mathrm{d}x^a \wedge \theta_a.$$

Hence T^*N has an almost-Kähler structure which is Kähler (J is integrable) iff (N, u) is locally flat.

Now, let us come back to S^*N. Clearly, TS^*N is defined in TT^*N by $\mathrm{d}(u^{ab}\xi_a\xi_b) = 0$, which is equivalent to

$$(3.16) \qquad u^{ab}\xi_a\theta_b = 0.$$

Hence $\mathcal{H}/S^*N \subset TS^*N$, and, on S^*N, we simply refer to this space as $\mathcal{H}$. Furthermore, let us denote by A the *fundamental vector field* of the contact structure of S^*N defined by $i(A)\eta = 1$, $i(A)\mathrm{d}\eta = 0$ [B1]. We see by a straightforward checking that

$$A = u^{ab}\xi_a X_b = \xi_a Y^a \in \mathcal{H}, \tag{3.17}$$

which proves that $\mathcal{H}$ is transversal to the contact distribution E, and that $K = E \cap \mathcal{H}$ is a n-dimensional distribution on S^*N satisfying $\eta = 0$. From (3.15) we see that it also satisfies $\mathrm{d}\eta = 0$, hence [B1] it defines a new Legendre foliation on S^*N, transversal to $\mathcal{C}^*$.

Moreover, it is clear that $K \perp \mathcal{C}^*$ with respect to the metric g of (3.14). But we also have from (3.12), (3.14), (3.17)

$$g(A, Z) = u^{ab}\xi_a g(X_b,\ \zeta^c X_c) = u^{ab}u_{bc}\xi_a\zeta^a = \xi_a\zeta^a = 0 \ \text{ for every } Z = \zeta^c X_c \in K,$$

since $\eta = 0$ on K. Hence $K \perp A$. Now, let us remark that (3.17) also implies

$$JA = \xi_a \frac{\partial}{\partial \xi_a} \perp S^*N. \tag{3.18}$$

Hence, since J and g are compatible, and $J\mathcal{H} = \mathcal{V}$, we get $JK = \mathcal{C}^*$.

Since J is compatible with Ω of (3.15) on T^*N, it follows from the described properties that $-J$ is a complex structure compatible with $\mathrm{d}\eta$ on E, and $g\,|\,E$ is the corresponding Hermitian metric, which is the first thing we need in a Maslov class computation.

Now, in order to go on with the computation, we need adequate connections as shown in Section 2. For the sake of simplicity, let us assume that u is a locally flat metric on N. In this case, g is a locally flat Kähler metric on T^*N, its Riemannian connection ∇ is J-compatible, and $\mathcal{H}$, $\mathcal{V}$ are ∇-parallel. The hypersurface S^*N has the normal JA in T^*N, hence we have the Gauss equation

$$\nabla_{X^*} Y^* = \nabla^*_{X^*} Y^* + b^*(X^*, Y^*)\, JA, \tag{3.19}$$

where the star means that we are refering to elements in S^*N, ∇^* is the *induced connection*, and b^* is the *second fundamental form*. From (3.19), and since $\mathcal{V}$ is ∇-parallel, we see immediately that $\mathcal{C}^*$ is ∇^*-parallel, and ∇^* induces on $\mathcal{C}^*$ on orthogonal connection. This may be used to define the connection $\overset{1}{\nabla}$ of (2.12) (if restricted to the Legendre submanifold M, of course).

Now, let us concentrate on the Legendre submanifold M of S^*N, and let (e_i) $(i = 1, \ldots, n)$ be a local g-orthonormal field of frames in TM. Then, since TM is lagrangian in E, and $A \perp E$, it follows that (Je_i, A) is a normal basis of M in S^*N, and we may write the Gauss equations of M is S^*N under the form

$$\nabla^* e_i = \mu_i^j e_j + c_i^j (Je_j) + \kappa_i A, \tag{3.20}$$

where (μ_i^j) are the local 1-forms of the connection ∇' induced by ∇^* in M, and c_i^j, κ_i are 1-forms which define the second fundamental form of M in S^*N. Since the induced connection ∇' preserves the metric induced by g in M, it is clear that we may use ∇' to get the connection $\overset{2}{\nabla}$ needed in (2.12).

In order to compute Maslov classes we have to find the difference between $\overset{2}{\nabla}$ and $\overset{1}{\nabla}$ expressed in common frames of reference, which we take to be the complex frames associated to (e_i) considered above in $(E, -J)$. These are defined by (2.1), which in our case is

$$\epsilon_i = \frac{1}{\sqrt{2}} (e_i + \sqrt{-1}\, Je_i) \tag{3.21}$$

(since our complex structure is $-J$), and the forms $\overset{2}{\pi}{}_i^j$ of (2.12) are precisely μ_i^j. We Still have to compute $\overset{1}{\pi}{}_i^j$ of the local equations $\overset{1}{\nabla}\epsilon_i = \overset{1}{\pi}{}_i^j \epsilon_j$ of our $\overset{1}{\nabla}$, and, by (3.21) this can be done if we first compute $\nabla^* (Je_i)$. Using (3.19), and the compatibility of ∇ and J, we may go on as follows

$$\begin{aligned} \nabla^*(Je_i) &= \nabla(Je_i) - \beta_i(JA) = J(\nabla e_i) - \beta_i(JA) = \\ &= J(\nabla^* e_i + \alpha_i(JA)) - \beta_i(JA) = \\ &= \mu_i^j (Je_j) - c_i^j e_j - \alpha_i A + (\kappa_i - \beta_i)(JA), \end{aligned} \tag{3.22}$$

where β_i and α_i are 1-forms defined by

$$\beta_i(X^*) = b^*(X^*, Je_i), \alpha_i(X^*) = b^*(X^*, e_i).$$

Now, since $\nabla^*(Je_i)$ is tangent to S^*N we must have $\kappa_i = \beta_i$, and

$$\nabla^*(Je_i) = \mu_i^j (Je_j) - c_i^j e_j - \alpha_i A. \tag{3.23}$$

(These are the «main part» of the Weingarten equations of M in S^*N. To have all of the Weingarten equations we should also write down a formula for $\nabla^* A$, but we are not interested in it. As a consequence of (3.21), (3.20), and (3.23), we have

$$\nabla^* \epsilon_1 = (\mu_i^j - \sqrt{-1}\, c_i^j)\, \epsilon_j + \frac{1}{\sqrt{2}} (\kappa_i - \sqrt{-1}\, \alpha_i)\, A. \tag{3.24}$$

Now let us be more precise about $\overset{1}{\nabla}$. Namely, $\overset{1}{\nabla}$ is the complex connection on $(E, -J)$ obtained by the extension of ∇^*/C^*, and it follows that $\overset{1}{\nabla}$ can also be seen as the connection induced by ∇^* on E (since the latter extends ∇^*/C^*). Hence, in view of (3.24) we have

(3.25) $\quad \overset{1}{\pi}{}_i^j = \mu_i^j - \sqrt{-1}\, c_i^j.$

Therefore

(3.26) $\quad \overset{2}{\pi}{}_i^j - \overset{1}{\pi}{}_i^j = \sqrt{-1}\, c_i^j,$

and we can use this result for the computation of the Maslov classes.

For the result that we are interested in, we use (2.15) and (2.16), and we have in view of (3.26)

(3.27) $$m^*(M) = -\frac{1}{\pi}\,[c_i^i]$$

(brackets denote the cohomology class). We shall reformulate this result such as to make clear its geometric significance.

It follows from (3.20) that the second fundamental form of M in S^*N is given by

(3.28) $$\sigma(X, Y) = [c_j^h(X)(Je_h) + \kappa_j(X)A]\eta^j,$$

where X, Y are in TM, and $Y = \eta^j e_j$. Accordingly, the *mean curvature vector* of M in S^*N is

(3.29) $$H = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i) = \frac{1}{n}\sum_{i=1}^{n} (c_i^h(e_i)(Je_h) + \kappa_i(e_i)A).$$

Since ∇^* preserves g, and by using (3.20), (3.23), we obtain

(3.30) $$0 = \mathrm{d}(g(e_i, Je_j)) = g(\nabla^* e_i, Je_j) + g(e_i, \nabla^* Je_j) = c_i^j - c_j^i,$$

and we may change in (3.29) $c_i^h(e_i)$ by $c_h^i(e_i)$. On the other hand, (3.20) yields

$$c_h^j(e_i) = g(\nabla^*_{e_i} e_h, Je_j),$$

and since ∇^* has no torsion we obtain $c_h^j(e_i) = c_i^j(e_h)$, and therefore H finally becomes

(3.31) $$H = \frac{1}{n}\sum_{h,i=1}^{n} c_i^i(e_h)(Je_h) + \frac{1}{n}\left(\sum_{i=1}^{n} \kappa_i(e_i)\right)A = H' + \frac{1}{n}\left(\sum_{i=1}^{n} \kappa_i(e_i)\right)A,$$

where H' denotes the orthogonal projection of H on E. Since $(e_i, -Je_i)$ is a symplectic basis for the symplectic form $\mathrm{d}\eta/E$, we have

(3.32) $$\mathrm{d}\eta = \sum_{k=1}^{n} \epsilon^k \wedge (\epsilon^k \circ J),$$

where $(\epsilon^k, \epsilon^k \circ J)$ is the dual cobasis, and a simple computation provides us with the final result

THEOREM. *The Maslov class of the Legendre submanifold M of S^*N, where N is a flat Riemannian manifold, is given by*

$$m^*(M) = \frac{n}{\pi} [i(H')d\eta]. \tag{3.33}$$

Particularly, if M is a minimal submanifold of S^*N, $m^*(M) = 0$. ∎

This theorem is analogous to Morvan's result [M], [D], [V_2],

Final Remarks. 1) We took N flat in order to simplify the exposition. As a matter of fact, $m^*(M)$ can be computed by formula (3.27) in the general case, but then we must replace ∇ by another connection which can be seen as a sum of the projections of ∇ on $\mathcal{H}$ and V (the *second connection* of the Riemannian manifold T^*N endowed with the foliation V, as defined in [V_1]). However, (3.33) cannot be obtained in the general case since this new connection has torsion.

2) The established machinery can also be used to compute higher dimensional Maslov classes by means of (2.12), (2.14), but we must then use also the curvature of M, and the Gauss-Codazzi integrability conditions, which makes the computation complicated, and with not very nice final results.

3) It is possible to transfer the above computation of the Maslov class to Legendre submanifolds of the tangent unit spheres bundle of a Riemannian manifold, by means of the machinery of the Legendre transformation [AM].

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Manuscript received: October 29, 1985.